

# Normalized Range Voting Broadly Resists Control\*

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April 26, 2011

## Abstract

We study the behavior of Range Voting and Normalized Range Voting with respect to electoral control. Electoral control encompasses attempts from an election chair to alter the structure of an election in order to change the outcome. We show that a voting system resists a case of control by proving that performing that case of control is computationally infeasible. Range Voting is a natural extension of approval voting, and Normalized Range Voting is a simple variant which alters each vote to maximize the potential impact of each voter. We show that Normalized Range Voting has among the largest number of control resistances among natural voting systems.

## 1 Introduction

Many of the key results in voting theory show that all voting systems are flawed in some way. Arrow’s Impossibility Theorem states that in any election with more than two candidates each voting system will violate at least one of several reasonable and natural criteria [Arr50]. The Gibbard–Satterthwaite Theorem and Duggan–Schwartz Theorem show that all reasonable voting systems are susceptible to strategic voting, where a voter may be able to vote counter to his or her true preferences and achieve a better outcome [Gib73, Sat75, DS00]. With any voting system, it might be possible for a dishonest election organizer able to subtly alter the election to achieve his or her desired end. Thus much of the following study of voting systems has been directed toward finding the best compromises and most reasonable, if imperfect, solutions.

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\*Supported in part by NSF grants CCF-0426761, IIS-0713061, and CCF-0915792. This work originally appeared as my masters thesis at the Rochester Institute of Technology. A version of this paper appears as University of Rochester technical report TR956.

Electoral control represents cases where the authority conducting the election attempts to alter the outcome by changing the structure of the election. The study of control of elections was initiated by Bartholdi, Tovey, and Trick [BTT92], who also introduced a novel defense against it. Even if control is possible, it may be computationally very difficult to find an ideal plan. The standard tools of complexity theory can be brought to bear on the problem. In many cases, a control problem can be shown to be NP-hard and thus very unlikely to be solvable in polynomial time. We may be able to accept theoretical vulnerability to control if computational difficulty would make it essentially impossible for any computationally limited attacker.

Since the initial work of Bartholdi, Tovey, and Trick, a number of voting systems have been studied with an eye toward computational resistance. Several systems have been found with a high number of resistances [ENR09, HHR09, FHHR09], although some of them are not sufficiently natural for practical use, remain vulnerable to some of the cases of control, or have other technical flaws. The system fallback voting was found by Erdélyi et al. [EFPR11] to be resistant to all but two cases of control and it possesses the best known set of resistances for a natural system. Voting systems have been developed that resist all cases of control, but at the cost of being highly unnatural [HHR09]. Thus it is still highly desirable to search for natural and robust voting systems with high degrees of control resistance.

This paper is particularly motivated by the work of Erdélyi et al. who studied the system sincere-strategy preference-based approval voting (SP-AV) [ENR09]. This system is a hybrid of approval voting and plurality, and it handily combines the control resistances of both systems. It does so by adding a vote coercion step that adjusts all votes to approve of and disapprove of at least one candidate. This results in more complex behavior upon changes to the candidate set and gives the system the candidate-control resistances possessed by plurality. However this may have the effect of forcing a voter to distinguish between candidates he or she ranked evenly and assigning him or her an arbitrary new approval threshold that may not represent his or her preferences.

Range voting is a voting system with an alternate voter preference representation that allows a voter to score his or her level of approval of each candidate [Smi00]. Range voting has a number of real world advocates due to its good behavior regarding conventional voting system criteria. We will also introduce a variant of range voting and show that it has among the highest degrees of resistance to control among natural voting systems, matching the set of control resistances possessed by Fallback voting. This system, normalized range voting, uses a similar vote-alteration procedure to SP-AV, but in a way that preserves the relative preferences of a voter among the candidates in the election.

## 2 Range Voting

Range voting (RV) is a voting system with an alternate voter representation that allows voters to express his or her degree of approval in each candidate. Let  $\|S\|$  denote the cardinality of a finite set  $S$ . We will describe a  $k$ -range election as  $E = (C, V)$  where  $C$  is the set of candidates with  $\|C\| = m$ , and  $V$  is the set of voters with  $\|V\| = n$  and for a voter  $v \in V$ ,  $v \in \{0, 1, \dots, k\}^m$ . Each voter expresses his or her preferences by giving a score for each candidate. The parameter  $k$  sets the highest score a voter is allowed to give a candidate. The winners of the election are the candidates with the highest sum score across all voters.

**Example** The following is an example of a 2-range election of the candidates  $\{a, b, c\}$ .  $a$  will be the winner with a total of 14 points.

# Voters	$a$	$b$	$c$
5	2	0	1
6	0	2	0
4	1	2	0

Though range voting is also sometimes described allowing scores over a real interval such as  $[0, 1]$  [Smi00], this paper will deal with the more limited integral version for its practicality of implementation and to avoid issues with the size of representation. Our primary concern is to study the difficulty of decision problems relating to the system and allowing scores of unbounded size would greatly complicate that analysis. Note that any bounded size and precision real number representation would be equivalent to an integral representation, so this version will be just as expressive as a rational representation or any other which would be suitable for computational analysis.

Arrow's Theorem was formulated with the traditional voter preference models of a strict ordering. Since range voting uses a different model, it is not bound by that result and, though subject to interpretation, achieves all of the criteria, which is normally impossible for a voting system [Smi00, Hil05].

To demonstrate, we will show an example where RV satisfies independence of irrelevant alternatives (IIA) while plurality would violate it. IIA is satisfied in an election system if the relative ranking between two candidates is independent of the presence or ranking of other candidates. Let us formulate a 1-range election, and assume that each voter only gives any points to his or her top candidate.

# Voters	$a$	$b$	$c$	Ranked ballot
5	1	0	0	$a > b > c$
4	0	1	0	$b > a > c$
2	0	0	1	$c > b > a$

Candidate  $a$  wins this initial election in either election system. Now, if we remove the last place candidate  $c$ :

# Voters	$a$	$b$	Ranked Ballot
5	1	0	$a > b$
4	0	1	$b > a$
2	0	0	$b > a$

In the range election, the  $c$  voters are left not awarding any points to anybody, which is a perfectly legal vote, and perfectly rational, if one does feel no distinction between the candidates. Consequently the original result stands and  $a$  remains the winner. In the plurality election, the votes of third group of voters are transferred to their second choice and  $b$  becomes the new winner of the election.

### 3 Normalized Range Voting

A rational voter seeking to maximize his or her impact in an election would always give his or her most preferred candidate the highest score possible ( $k$ ) and his or her least preferred candidate the lowest score possible (0). We introduce the system Normalized Range Voting (NRV), which captures this and also gives the system more interesting behavior under several types of centralized control.

In this system each voter specifies his or her preferences as in standard range voting. However, as part of the score aggregation, the system normalizes each vote to a rational in the range  $[0, k]$ . Formally, for a voter  $v$  and  $v$ 's maximum and minimum scores  $m$  and  $n$ , his or her score  $s$  for a candidate is changed to  $\frac{k(s-n)}{m-n}$ . If  $m = n$ , a voter shows no preference among the candidates and this vote will not be counted. The system does not make an effort to coerce such an unconcerned vote into one that distinguishes between the candidates.

The relationship between RV and NRV is closely analogous to the relationship between approval voting and SP-AV. The normalization step ends up removing several cases of control immunity, but it introduces more complex behavior on alterations of the candidate set that gain back a greater number of control resistances.

Unlike RV, NRV unambiguously fails the criteria independence of irrelevant alternatives. Consider a 2-NRV election with  $C = \{a, b, c\}$  and  $V$  below.

# Voters	$a$	$b$	$c$
7	2	0	0
4	0	2	0
4	0	1	2

$a$  will win this election with a score of 14, with 12 and 8 for candidates  $b$  and  $c$ . However, consider the election with the same voters but with the candidate  $c$  removed.

# Voters	$a$	$b$
7	2	0
4	0	2
4	0	1

At first,  $a$  appears to still be winning the election. However the normalization step will scale up the votes from the third group of voters to give  $b$  16 points in total, making  $b$  the winner of the election.

While this seems to be a negative against this system, this complex, shifting behavior on the changing of the candidates is exactly what allows us to achieve a large number of control resistances for NRV over RV.

## 4 Control

Control represents the efforts of a centralized authority, the chair of an election to alter the structure of the election in order to affect its outcome. This involves changing either the candidate or voter sets or partitioning either into subelections. In real world political elections, this corresponds to voter fraud and voter suppression, back-room dealings with potential candidates, and gerrymandering and similar manipulations. In the context of multiagent systems, it is related to any efforts by a system designer or administrator to alter the results by changing the parameters of the system.

More formally, for the purposes of the complexity theoretic analysis of the control problems, we will analyze the cases of control in the form of decision problems. That is, we will define a problem where the goal is to find whether in a particular election a certain case of control can succeed in its goals.

The goal is to classify a voting system as vulnerable, resistant, or immune to each of the various cases of control, with these terms initially defined by Bartholdi et al. [BTT92] and widely adopted since. It is helpful now to define these notions precisely.

**Vulnerability** A voting system is *vulnerable* to a case of control if that action has potential to affect the result of an election, and the associated decision problem can be solved in polynomial time; that is, it is in P. This has a very good practical correspondence with real world efficiency of the problem, and thus the case of control is computationally easy.

**Resistance** A voting system is *resistant* to a case of control if that action has potential to affect the result of an election, and the associated decision problem is NP-hard. The idea of NP-hardness has a long and storied history, but for the current purposes, it suffices to say that such problems are very unlikely to have efficient solutions, barring a major shift in our understanding of computer science.

**Immunity** A voting system is *immune* to a case of control if that action cannot affect the result of the election. This is obviously a desirable notion but it is generally harder to come by, and many immunities are incompatible with very basic and reasonable properties of voting systems [ENR09].

The control cases of Bartholdi et al. were all *constructive*, that is, the control is directed towards making a distinguished candidate the winner. In some cases, a malicious chair could conceivably want above all to prevent a particular candidate from winning the election, regardless of who else wins. This idea was introduced by Conitzer et al. as *destructive* manipulation and later by Hemaspaandra et al. in the context of control [HHR07]. Though this may seem to be a less desirable goal, it may be feasible in some cases where constructive control is not and thus is it also worth studying.

Among the cases of control are control by adding or deleting either voters or candidates. In the case of adding voters or candidates, the new participants must be chosen from a set rather than arbitrarily created. While this type of control is not necessarily thus limited, the decision problems are defined here as having a limit on the number of voters or candidates that can be added or deleted. In the candidate cases, the distinguished candidate must be in the original candidate set. In the cases of destructive control by deleting candidates, the distinguished candidate cannot be among those deleted as that would trivially solve the problem.

The various cases of control by partition are not quite straightforward and deserve a little explanation. In any control by partition problem, initial subelections are performed with segments of the voter and candidate sets and a final election is performed with the candidates that survive these subelections.

In control by partition of voters, the voter set is partitioned into two subsets and an subelections are run with each (with the original candidate set). The candidates that survive each subelection face off to find the final winner of the election.

Control by partition of candidates has two major variants. In one variant, *control by partition*, one set of candidates is separated off from the rest for an initial subelection. Whatever candidates survive this election then rejoin the rest of the candidates for the final election with the entire voter set. In the other variant, *control by run-off partition*, the candidate set is partitioned into two sets and each set conducts an initial subelection. The candidates that survive each of these elections then are brought together for the final election with the entire voter set.

There is an additional variation in the tiebreaking rule that is chosen in the subelections. In the case of a tie, either all of the top scoring candidates are promoted to the final election, or none of them are. These two cases are called *ties-promote* and *ties-eliminate*. Notably, in the second case, an election can fail to elect any candidate. Though these may seem like subtle differences, many voting systems will resist one of the cases while being vulnerable to another.

## Control by Adding Candidates

**Given** An election  $E = (C, V)$ , a distinguished candidate  $w \in C$ , a spoiler candidate set  $D$ , and  $k \in \mathbb{N}^+$

**Question (Constructive)** Is it possible to make  $w$  the winner of an election  $(C \cup D', V)$  with some  $D' \subseteq D$  where  $|D'| \leq k$ ?

**Question (Destructive)** Is it possible to make  $w$  not the winner of an election  $(C \cup D', V)$  with some  $D' \subseteq D$  where  $|D'| \leq k$ ?

## Control by Deleting Candidates

**Given** An election  $E = (C, V)$ , a distinguished candidate  $w \in C$ , and  $k \in \mathbb{N}^+$

**Question (Constructive)** Is it possible to make  $w$  the winner of an election  $(C - C', V)$  with some  $C' \subseteq C$  where  $|C'| \geq k$ ?

**Question (Destructive)** Is it possible to make  $w$  not the winner of an election  $(C - C', V)$  with some  $C' \subseteq (C - \{w\})$  where  $|C'| \leq k$ ?

## Control by Adding Voters

**Given** An election  $E = (C, V)$ , a distinguished candidates  $w \in C$ , an additional voter set  $U$ , and  $k \in \mathbb{N}^+$

**Question (Constructive)** Is it possible to make  $w$  the winner of an election  $(C, V \cup U')$  for some  $U' \subseteq U$  where  $|U'| \leq k$ ?

**Question (Destructive)** Is it possible to make  $w$  not the winner of an election  $(C, V \cup U')$  for some  $U' \subseteq U$  where  $|U'| \leq k$ ?

## Control by Deleting Voters

**Given** An election  $E = (C, V)$ , a distinguished candidates  $w \in C$ , and  $k \in \mathbb{N}^+$

**Question (Constructive)** Is it possible to make  $w$  the winner of an election  $(C, V - V')$  for some  $V' \subseteq V$  where  $|V'| \leq k$ ?

**Question (Destructive)** Is it possible to make  $w$  not the winner of an election  $(C, V - V')$  for some  $V' \subseteq V$  where  $|V'| \leq k$ ?

## Control by Partition of Candidates

**Given** An election  $E = (C, V)$  and a distinguished candidates  $w \in C$

**Question (Constructive)** Is there a partition  $C_1, C_2$  of  $C$  such that  $w$  is the final winner of the election  $(D \cup C_2, V)$ , where  $D$  is the set of candidates surviving the initial subelection  $(C_1, V)$ ?

**Question (Destructive)** Is there a partition  $C_1, C_2$  of  $C$  such that  $w$  is not the final winner of the election  $(D \cup C_2, V)$ , where  $D$  is the set of candidates surviving the subelection  $(C_1, V)$ ?

## Control by Runoff Partition of Candidates

**Given** An election  $E = (C, V)$  and a distinguished candidates  $w \in C$

**Question (Constructive)** Is there a partition  $C_1, C_2$  of  $C$  such that  $w$  is the final winner of the election  $(D_1 \cup D_2, V)$ , where  $D_1$  and  $D_2$  are the sets of surviving candidates from the subelections  $(C_1, V)$  and  $(C_2, V)$ ?

**Question (Destructive)** Is there a partition  $C_1, C_2$  of  $C$  such that  $w$  is the final winner of the election  $(D_1 \cup D_2, V)$ , where  $D_1$  and  $D_2$  are the sets of surviving candidates from the subelections  $(C_1, V)$  and  $(C_2, V)$ ?

## Control by Partition of Voters

**Given** An election  $E = (C, V)$  and a distinguished candidates  $w \in C$

**Question (Constructive)** Is there a partition  $V_1, V_2$  of  $V$  such that  $w$  is the final winner of the election  $(D_1 \cup D_2, V)$  where  $D_1$  and  $D_2$  are the sets of surviving candidates from the subelections  $(C, V_1)$  and  $(C, V_2)$ ?

**Question (Destructive)** Is there a partition  $V_1, V_2$  of  $V$  such that  $w$  is not the final winner of the election  $(D_1 \cup D_2, V)$  where  $D_1$  and  $D_2$  are the sets of surviving candidates from the subelections  $(C, V_1)$  and  $(C, V_2)$ ?

## 5 Results

The control results for these two systems, as well as approval and SP-AV for comparison, are summarized in Table 1. “V”, “I”, and “R” stand for vulnerable, immune, and resistant, which are used in the standard way in the literature dating from Bartholdi, Tovey, and Trick [BTT92]. “C” marks constructive control while “D” marks destructive control.



Control by	Tie Model	Approval		SPAV		Fallback		RV		NRV	
		C	D	C	D	C	D	C	D	C	D
Adding candidates		I	V	R	R	R	R	I	V	R	R
Deleting candidates		V	I	R	R	R	R	V	I	R	R
Partition of Candidates	TE	V	I	R	R	R	R	V	I	R	R
	TP	I	I	R	R	R	R	I	I	R	R
Run-off Partition of Candidates	TE	V	I	R	R	R	R	V	I	R	R
	TP	I	I	R	R	R	R	I	I	R	R
Adding Voters		R	V	R	V	R	V	R	V	R	V
Deleting Voters		R	V	R	V	R	V	R	V	R	V
Partition of Voters	TE	R	V	R	V	R	R	R	V	R	R
	TP	R	V	R	R	R	R	R	V	R	R

Table 1: Control Results for Approval, Sincere-Strategy Preference-Based Approval Voting [ENR09], Fallback Voting [EFPR11], Range Voting, Normalized Range Voting

## 5.1 Generalization of Resistance Results

The proofs here will refer necessarily to specific RV and NRV elections with a particular scoring range  $k$ , and additionally they will use different values as necessary. However we want to be able to show resistance for other scoring ranges, and show that all the resistances we show will hold for some particular scoring range.

**Theorem 5.1** *If RV or NRV exhibits resistance to a case of control for a particular scoring range  $k$ , it will exhibit that resistance for any range  $ak$  with  $a \in \mathbb{N}^+$ .*

**Proof** We can reduce an instance of any RV or NRV control problem for an election with a scoring range  $k$  to an instance of the same problem with an election with a scoring range of  $ak$  for any  $a \in \mathbb{N}^+$ . We can do this simply by scaling all of the scores in all of the votes in the original election up by a factor of  $a$ . This new election with the new scoring bound and new votes will behave the same as the original election before any control attempt, and it will also behave the same under any control attempt or manipulative action, as all of the votes and the sum scores will be scaled up by the same factor  $a$ . In the case of NRV, any normalization that occurs in the original election will occur in the newly scaled election to the same degree, but just with the pre and post normalization scores both being scaled up by the factor of  $a$ . Thus the winner in the scaled election will be the same before and after any control action and control problems easily reduce to same problem in the scaled voting system.  $\square$

## 5.2 Results Derived From Approval

Due to RV's great similarity with approval voting, many results relating to approval trivially apply to RV and NRV.

**Theorem 5.2** *If approval voting is resistant to a case of control, RV and NRV will also be resistant for any scoring range.*

This is easy to show. We can reduce from an instance of any approval control problem by simply considering the election a 1-range election or a 1-normalized-range election. A 1-range election is exactly equivalent so this will trivially work. For the NRV election, though this does technically include the normalization step which can modify the election, when the score range is 1, no normalization is actually performed, so again this election is equivalent to the original approval election. These results will also generalize to  $k$ -RV and  $k$ -NRV any  $k \geq 1$  as previously described.

**Theorem 5.3** *1-RV and 1-NRV are resistant to the following cases of control: constructive control by adding voters, constructive control by deleting voters, and constructive control by the partition of voters in the ties-promote and ties-eliminate models.*

All of these resistances are derived from reductions from approval as described above, and the fact that approval is resistant to these cases of control [HHR07].

### 5.3 Adding/Deleting Candidates

**Theorem 5.4** *2-NRV is resistant to constructive control by adding or deleting candidates and destructive control by adding candidates.*

**Proof.** This proof is inspired by a similar proof relating in SP-AV by Erdélyi, Nowak, and Rothe [ENR09].

We will reduce from an instance of the hitting set problem, defined as follows [GJ79].

**Given:** A collection  $\mathcal{S}$  of subsets of a set  $B$ ,  $k \in \mathbb{N}^+$

**Question:** Does  $B$  contain a hitting set  $B'$  of size  $k$  or less that contains at least one element from every  $S \in \mathcal{S}$ ?

Given a hitting set instance  $(B, (\mathcal{S}), ||)$  with  $|B| = n$  and  $|f| = m$  we will construct a 2-range election. The candidate set  $C$  will consist of  $B \cup \{c, w\}$ . The idea is that  $c$  will win the election unless only a hitting set of size  $k$  of candidates from  $B$  are included. The voter set  $V$  will be as follows:

- $2m(k + 1) + 4n$  voters have a score of 2 for  $c$ , and a score of 0 for all other candidates.
- $3m(k + 1) + 2k + 1$  voters have a score of 2 for  $w$ , and a score of 0 for all other candidates.

- For each  $b \in B$ , 4 voters have a score of 2 for  $b$ , a score of 1 for  $w$ , and a score of 0 for all other candidates.
- For each  $S_i \in \mathcal{S}$ ,  $2(k+1)$  voters have a score of 2 for  $b$ , for each  $b \in S_i$ , a score of 1 for  $c$ , and a score of 0 for all other candidates.

This will lead to scores in  $(\{c, w\}, V)$  as follows:

Candidate	Score
$c$	$8m(k+1) + 8n$
$w$	$6m(k+1) + 8n + 4k + 2$

The candidate  $c$  will win with a margin of  $2m(k+1) - 4k - 2$ . Additionally the scores in  $(\{c, w\} \cup B, V)$  will be as follows:

Candidate	Score
$c$	$6m(k+1) + 8n$
$w$	$6m(k+1) + 4n + 4k + 2$
$b \in B$	$\leq 8 + 4m(k+1)$

Here,  $c$  will win with a margin of  $4n - 4k - 2$ , which will be positive as long as  $k < n$ .

We will show that  $w$  will be the winner of  $(\{c, w\} \cup B', V)$  with  $B' \subseteq B$  if  $B'$  corresponds to a hitting set of size  $\leq k$ . The candidate  $w$  loses 4 points for each  $b \in B'$  included, of which there are no more than  $k$ .  $c$  loses  $2(k+1)$  points for each  $S_i$  hit. There will be  $m$  such sets if  $B'$  is a hitting set, so  $c$  loses  $2m(k+1)$  points total.

Candidate	Score
$c$	$6m(k+1) + 8n$
$w$	$\geq 6m(k+1) + 8n + 2$
$b \in B'$	$\leq 8 + 4m(k+1)$

$w$  will end up with an advantage of at least 2 points and thus  $w$  will be the winner of the election.

We will show that  $w$  will not be the winner of any election  $(\{c, w\} \cup B', V)$  where  $B'$  does not correspond to a hitting set of size  $\leq k$ . If  $B'$  is a hitting set but  $|B'| > k$ ,  $c$  will have  $6m(k+1) + 8n$  points and  $w$  will have  $\leq 6m(k+1) + 8n$ . If  $B'$  is not a hitting set  $c$  will have  $\geq 6m(k+1) + 8n + 2(k+1)$  points and  $w$  will have  $\leq 6m(k+1) + 8n + 2k + 2$  points. In either case  $w$  will not be the unique winner.

This construction can be used to create cases of constructive and destructive control by adding candidates and destructive control by deleting candidates.  $((C, V), (m-k), c)$  is such an instance of destructive control by deleting candidates.  $((\{c, w\}, V), B, k, c)$  is an instance of destructive control by adding candidates, and  $((\{c, w\}, V), B, k, w)$  is an instance of constructive control by adding candidates.  $\square$

**Theorem 5.5** *2-NRV is resistant to destructive control by deleting candidates.*

As in Erdélyi, Nowak, and Rothe [ENR09], the previous reduction is not sufficient to show resistance to constructive control by deleting candidates, as  $c$  and  $w$  are the only candidates with a shot at winning. Deleting  $c$  will instantly make  $w$  the winner. The remaining case can be handled by the following reduction.

**Proof** As in the proof of Theorem 5.4, we reduce from an instance of hitting set  $(B, \mathcal{S}, k)$ .

The candidate set  $C$  will consist of  $B \cup \{w\}$ , the set  $B$  from the instance of hitting set together with an additional candidate.

The voter set  $V$  will be constructed as follows:

- $n + k$  voters have a score of 2 for  $b$  for every  $b \in B$ , and a score of 0 for  $w$ .
- $3 + 2mk$  voters have a score of 2 for  $w$  and a score of 0 for all other candidates.
- For each  $S \in \mathcal{S}$ ,  $4k + 1$  voters have a score of 2 for  $s$  for every  $s \in S$ , a score of 1 for each candidate in  $B - S$ , and a score of 0 for  $w$ .
- For each  $S \in \mathcal{S}$ ,  $4k + 1$  voters have a score of 2 for  $b$  for every  $b \in B - S$ , a score of 2 for  $w$ , and a score of 1 for  $s$  for every  $s \in S$ .
- For each  $b \in B$ ,  $2n - k$  voters have a score of 2 for  $b$ , a score of 1 for  $w$ , and a score of 0 for every other candidate.

We can show that if  $B'$  is a hitting set of size  $k$ ,  $w$  will be the winner of the election  $(B' \cup \{w\}, V)$ . Assume  $B'$  is a hitting set and  $\|B'\| = k$ . Each  $b \in B'$  will receive  $12mk + 4n - 2k + 4$  points.  $w$  will receive  $8mk + 6 + 4mk + 4(n - k) + 2k = 12mk + 4n - 2k + 6$  points. Thus  $w$  will be the winner in the election.

We can show that if  $B'$  is not a hitting set or if  $\|B'\| > k$ ,  $w$  will not be the winner of the election  $(B' \cup \{w\}, V)$ . First assume  $B'$  is a hitting set but  $\|B'\| = l > k$ . Since  $B'$  is a hitting set, every  $b \in B'$  will receive exactly  $12mk + 4n - 2k + 4$  points.  $w$  will receive  $4mk + 6 + 8mk + 4(n - l) + 2l = 12mk + 4n - 2l + 4$  points.  $\text{score}(b) - \text{score}(w) = -2k + 2l - 2$  which is non-negative since  $l > k$ . Thus  $w$  will lose the election.

Next consider the case where  $\|B'\| = l \leq k$  but  $B'$  is not a hitting set. Thus every  $b \in B'$  will have a score  $\geq 12nk + 4m + 2k + 4$  as each will gain an extra  $4k$  points from one set of group 3 voters.  $w$  will have the score  $12nk + 4m - 2l + 6$ . Thus  $\text{score}(b) - \text{score}(w) = 2k + 2l - 2$  which is non-negative and  $w$  will again lose the election.

An instance of hitting set  $(B, \mathcal{S}, k)$  can thus be reduced to finding whether  $w$  can be made the winner of  $(C, V)$  as above by deleting  $m - k$  candidates.  $\square$

## 5.4 Destructive Control by Partition of Voters

**Theorem 5.6** *2-NRV is resistant to destructive control by partition of voters in the ties-promote model.*

**Proof** We will reduce from restricted hitting set. Restricted hitting set is an NP-complete hitting set variant introduced by Hemaspaandra, Hemaspaandra, and Rothe with additional restrictions on the sizes of the sets in an instance [HHR07]. The version as used here has a slightly stronger bound that is necessary due to the somewhat larger numbers required in this proof.

**Given:** A collection  $\mathcal{S}$  of subsets of a set  $B$ ,  $k \in \mathbb{N}^+$ , with  $|\mathcal{S}| = m$ ,  $|B| = n$ , and the additional restriction that  $m(k+1) + 3 \leq n - k$

**Question:** Does  $B$  contain a hitting set  $B'$  of size  $k$  or less that contains at least one element from every  $S \in \mathcal{S}$ ?

Given an instance of restricted hitting set  $(B, \mathcal{S}, k)$  with  $|B| = n$  and  $|\mathcal{S}| = m$ , create a 2-normalized-range election with  $C = B \cup \{w, c\}$  and  $V$  as follows.

- $2m(k+1) + 4n$  voters have a score of 2 for  $c$ , and a score of 0 for every other candidate.
- $3m(k+1) + 2k$  voters have a score of 2 for  $w$ , and a score of 0 for every other candidate.
- For each  $b \in B$ , 4 voters have a score of 2 for  $b$ , a score of 1 for  $w$ , and a score of 0 for every other candidate.
- For each  $S_i \in \mathcal{S}$ , for each  $b \in S_i$ ,  $2(k+1)$  voters have a score of 2 for  $b$ , a score of 1 for  $c$ , and a score of 0 for every other candidate.
- For each  $b \in B$ , 1 voter has a score of 2 for  $b$  and a score of 0 for every other candidate.

The candidate  $c$  can be made to lose  $(C, V)$  through partition of voters if and only if there is a hitting set of size  $\leq k$  over  $\mathcal{S}$  in  $B$ .

We can show that if there is a hitting set of size  $\leq k$ , it is possible to cause  $c$  to lose the election through partition of voters. Given an appropriate hitting set  $B'$ , partition  $V$  into sets  $V_1$  and  $V_2$ . Let  $V_1$  contain a voter from the final group corresponding to every  $b \in B'$  and one voter from the second group (allotting just 2 points to  $w$ ) and let  $V_2 = V - V_1$ . After the initial subelections, we will be left with  $w$ ,  $c$ , and the candidates  $B'$  corresponding to the hitting set, and  $w$  will win this election (see the reduction to adding/deleting candidates for the details of that proof).

If there is no hitting set  $B' \in B$  of size  $\leq k$ ,  $c$  cannot be made to lose the election through partition of voters. For any actions attempting to control the election by forcing the final candidate set, see the previous reduction to adding/deleting candidates. As for other efforts concentrated at more typically partitioning the voters, among the initial candidates,  $c$  has as high of a score as any two other candidates, so  $c$  must at least tie in at least one of the subelections. Thus he or she will always make it to the final election. The scores of the candidates in the initial election follow.

Candidate	Score
$c$	$6m(k+1) + 8n$
$w$	$6m(k+1) + 4k + 4n$
$b$	$\leq 4m(k+1) + 10$

$c$ 's score minus the next two highest scores will thus be at least  $4(n-k) - 4m(k+1) - 10$ . However, due to our use of restricted hitting set, we have that  $m(k+1) + 3 \leq n - k$ , and so this is at least 2. Thus the only way to defeat  $c$  is to face them against a hitting set of candidates as described.  $\square$

**Theorem 5.7** *4-NRV is resistant to destructive control by partition of voters in the ties-eliminate model.*

We will reduce from the Exact Cover by Three-sets problem (X3C), defined as follows.

**GIVEN** A set  $B = \{b_1, \dots, b_{3k}\}$  and a family  $\mathcal{S} = \{S_1, \dots, S_n\}$  of sets of size three of elements from  $B$ .

**QUESTION** Is it possible to select  $k$  sets from  $\mathcal{S}$  such that their union is exactly  $B$ ?

*Proof.*

Given a X3C instance  $B, \mathcal{S}$  we will construct a 2-range election  $(C, V)$  as follows. The candidate set will be  $B \cup \{c, w\}$ , where  $c$  will be the distinguished candidate. The voters set  $V$  will consist of the following:

- For every  $S_i \in \mathcal{S}$ , one voter with a score of 4 for every candidate in  $B - S_i$ , a score of 2 for  $c$ , and a score of 0 for every other candidate;
- $2n$  voters with a score of 4 for every candidate in  $B$ , a score of 2 for  $c$ , and a score of 0 for  $w$ ;
- $k - 1$  voters with a score of 4 for  $w$ , a score of 2 for  $c$ , and a score of 0 for all other candidates;
- For every  $b \in B$ , 1 voter with a score of 4 for  $b$ , a score of 1 for every candidate in  $B - b$ , a score of 1 for  $c$ , and a score of 0 for  $w$

- $2k + 3n + 1$  voters with a score of 4 for  $w$  and a score of 0 for every other candidate.

We will assume that  $1 \leq k \leq n$  and that each element in  $B$  is in at least one set from  $\mathcal{S}$ .

If there is an exact cover over  $B$ , then  $w$  can be made to lose the election through partition of voters in the ties-eliminate model. Consider the partition of the voter set into  $V_1, V_2$ , where  $V_1$  consists of the voters from the first group corresponding to the elements of the set cover together with the third group of voters, and where  $V_2 = V - V_1$ .

The candidate  $w$  will win the subelection  $(C, V_2)$  and the distinguished candidate  $c$  will easily win the subelection  $(C, V_1)$ .  $c$  will receive  $2k$  points from the cover voters and  $2k - 2$  points from the other voters for a total of  $4k - 2$  points. For the  $B$  candidates, each will receive 4 points from each of the first-group voters except the one that corresponds to the set that covers them, coming to  $4k - 4$  points in total.  $w$  will gain 4 points from each of the  $k - 1$  voters that favors them, and so he or she will gain  $4k - 4$  votes in total. Therefore  $c$  will win this subelection and will go on to face  $w$  in the final election.

In the final election, with almost all of the candidates eliminated, vote normalization will occur benefiting  $c$  to the point that he or she gains the advantage. All of the votes in the first, second, and fourth groups will now give four points to  $c$ , giving them  $12n + 14k - 2$  points in total. This is at least as many as the  $12n + 12k$  points  $w$  was given, and so  $w$  will no longer be the unique winner.

If there is no exact cover,  $w$  cannot be made to lose the election through partition of voters in the ties-eliminate model. This is due to several facts:  $w$  will win at least one of the two subelections,  $c$  cannot win either subelection, and  $w$  will win head to head against all candidate that it may face in the final election.

$c$  cannot win an initial subelection except as previously described. No voter prefers them outright, so the only way to make them win is to balance the points he or she gains from the first group of voters and from the third group of voters and to give he or she an advantage over each  $B$  candidate by covering that candidate with a voter that prefers  $c$ . None of the other voters will help  $c$  win a subelection, as no others help  $c$  gain points relative to the  $B$  candidates. If a set of first group voters smaller than  $k$  that is not a cover is chosen, then at least one  $B$  candidate will gain points for every one of these voters. We will then not be able to boost  $c$  over the  $B$  candidates without giving too many points to  $w$ . If a cover larger than  $k$  of voters is chosen, then there will not be enough third group voters to include to boost  $c$  over the  $B$  voters and  $c$  will not be able to win.

The candidate  $w$  must win at least one of the initial subelections and make it to the final election. With the original candidate set and unnormalized scores,  $w$  has a considerable advantage in points over all other candidates, and since it is largely the same voters that support all of the other candidates, there is no way to partition the voters to make  $w$  lose both subelections.

Against all candidates but  $c$ ,  $w$  will win a head to head contest in the final election. In a head-to-head contest with  $w$  the other candidates gain about as many points as  $c$  through normalization, but they will still have fewer points total as each  $B$  candidate loses out on 4 points for every subset  $S_i$  they are a part of. The  $B$  candidate will thus have no more than  $12n + 12k - 4$  votes, while  $w$  will have  $12n + 12k$  votes and will be the winner.

Therefore  $w$  will win the final election for all partitions in the case that there is not an exact cover over the set  $B$ .  $\square$

## 5.5 Partition of Candidates

**Theorem 5.8** *4-NRV is resistant to constructive control by partition and runoff partition of candidates.*

We will reduce from control by deletion of candidates in NRV. We will show the reduction to constructive control by run-off partition, though the other partition case is quite similar.

**Proof** Given an  $r$ -NRV election<sup>1</sup>  $(C, V)$  with  $|C| = m$  and  $|V| = n$  the distinguished candidate  $w \in C$ , and a deletion limit  $k \in \mathbb{N}$ , construct a  $2r$ -NRV election  $(C', V')$  as follows.  $C' = C \cup \{a, b\}$ , where  $a$  and  $b$  are additional auxiliary candidates.  $V'$  will consist of the original voter set  $V$  in addition to the following voters.

- For each  $c \in C$ ,  $2n$  voters have a score of  $2r$  for  $c$ , a score of  $r$  for  $a$ , and a score of 0 for every other candidate.
- For each  $c \in C^*$ ,  $3nm$  voters have a score of  $2r$  for  $c$  and a score of 0 for every other candidate.
- $2nm$  voters have a score of  $2r$  for  $w$ , a score of  $r$  for  $a$ , and a score of 0 for every other candidate.
- $nm$  voters have a score of  $2r$  for  $w$  and a score of 0 for every other candidate.
- $(m - k - 1)n$  voters have a score of  $2r$  for all  $c \in C$  and a score of 0 for every other candidate.
- $2n + 1$  voters have a score of  $2r$  for  $a$  and a score of 0 for every other candidate.
- $3n + 3nm + (m - k - 1)n + 2$  voters have a score of  $2r$  for  $b$  and a score of 0 for every other candidate.

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<sup>1</sup>Note that since  $k$ -NRV is resistant to deletion of candidates for  $k \geq 2$ , this reduction shows resistance for  $k \geq 4$ .



Let  $s_0(c)$  be the score of candidate  $c$  among the original voters  $V$ . Note for any candidate  $s_0(c) \leq nr$ .

The following are the scores for the candidates in  $(C', V')$  and various relevant subelections thereof.

$(C', V')$

Candidate	Score
$a$	$4nmr + 4nr + 2r$
$b$	$6nr + 6nmr + 2(m - k - 1)nr + 4r$
$c \in C^*$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(c)$
$w$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(w)$

The winner in this case will be  $b$ .

$(\{a, w\}, V')$

Candidate	Score
$a$	$4nr + 6nmr + 2$
$w$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(w)$

The winner of this election will be  $w$ .

$(\{w\} \cup D, V')$  where  $D \subseteq C^*$ ,  $|D| = l$

Candidate	Score
$c \in D$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(c)$
$w$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(w)$

The winner will again be whatever candidate is the winner over the original voter set  $V$ .

$(\{a, b\} \cup D, V')$  where  $D \subseteq C^*$ ,  $|D| = l$

Candidate	Score
$a$	$4nr + 6nmr + 2(m - l)nr + 2r$
$b$	$6nr + 6nmr + 2(m - k - 1)nr + 4r$
$c \in D$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(c)$

In this election,  $a$  will be the winner whenever  $l \leq k$ . Otherwise the winner will be  $b$ .

$(\{b, w\}, V')$

Candidate	Score
$b$	$6nr + 6nmr + 2(m - k - 1)nr + 4r$
$w$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(w)$

In this case,  $b$  is the clear winner.

$(\{a, w\} \cup D, V')$  where  $D \subseteq C^*, |D| = l$

Candidate	Score
$c \in D$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(c)$
$a$	$4nr + 4nmr + 2(m - l)nr + 2$
$w$	$4nr + 6nmr + 2(m - k - 1)nr + 2s_0(w)$

The winner will again be whatever candidate is the winner over the original voter set  $V$ .

We can show that if  $w$  can be made the winner of  $(C, V)$  by deleting  $\leq k$  candidates,  $w$  can be made the winner of  $(C', V')$  through control by runoff partition of candidates. Suppose  $w$  can be made the winner of  $(C, V)$  through deleting  $\leq k$  candidates. Let  $D$  be the set of candidates which were deleted in the deletion problem. Partition the candidates into the subelections  $(D \cup \{a, b\}, V')$  and  $(C - D, V')$ .  $a$  will win the first subelection as shown above.  $w$  will win the second subelection, as it must if it is capable of winning with the candidates in  $D$  deleted. The final election will then come down to  $w$  and  $a$ , and as we see above,  $w$  will come out the victor. Alternately, in the non-runoff partition case, let the initial subelection be  $(D \cup \{a, b\}, V')$ , which  $a$  will win. The final election will come down to  $(\{a, w\} \cup C - D, V')$ , which  $w$  will win.

We can show that if  $w$  can be made the winner of  $(C', V')$  through control by runoff partition of candidates,  $w$  can be made the winner of  $(C, V)$  by deleting  $\leq k$  candidates. Suppose  $w$  can be made the winner of the election  $(C', V')$  through control by runoff partition of candidates. It must be that this occurs through a partition of the form  $(\{a, b\} \cup D, \{w\} \cup (C^* - D))$  with  $D \subseteq C^*, |D| \leq k$ .  $b$  will always beat  $w$ , so they cannot face each other in either the initial or final elections. The only candidate capable of beating  $b$  is  $a$  when not in an election with  $w$  and when accompanied by no more than  $k$  other candidates from  $C$ .  $w$  must also be able to defeat the remaining  $m - k$  candidates from the initial election. Consequently  $w$  can also be made the winner of  $(C, V)$  by deleting  $k$  candidates.  $\square$

The preceding construction will shows that NRV is resistant to constructive cases of partition of candidates. However it is not sufficient for the destructive cases, as a winning candidate in the original election  $(C, V)$  will not actually win in  $(C', V')$ . Thus we will present a new construction to handle the destructive cases.

**Theorem 5.9** *2-NRV is resistant to destructive control by partition and runoff partition of candidates.*

**Proof** We can reduce the hitting set problem to the problem of destructive control by partition of candidates. Let  $(B, \mathcal{S}, k)$  be an instance of hitting set where  $B = \{b_1, b_2, \dots, b_n\}$ ,  $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ ,  $S_i \subseteq B$ , and  $k \in \mathbb{N}^+, k \leq n$ .

We will construct a 2-range election based on this instance. The candidate set  $C$  will consist of  $B \cup \{w\}$ . The voter set  $V$  will be as follows.

- For each  $S \in \mathcal{S}$ ,  $4(k+1)$  voters have a score of 2 for each  $b \in S$  and a score 1 for  $w$
- For each  $S \in \mathcal{S}$ ,  $4(k+1)$  voters have a score of 2 for each  $b \in B, b \notin S$ , and a score of 0 for every other candidate.
- For each  $b \in B$ , 4 voters have a score of 2 for  $b$ , a score of 1 for each  $b' \in B, b' \neq b$ , and a score of 0 for  $w$
- $2(k+1)m + 4n - 2k + 1$  voters have a score 2 for  $w$  and a score of 0 for every other candidate.

Again, we will first consider the outcome of several forms of subelections of this election.

$(C, V)$

Candidate	Score
$w$	$8(k+1)m + 8n - 4k + 2$
$b \in B$	$8(k+1)m + 4n + 4$

$w$  will win this election for any  $k \leq n$ .

$(\{w\} \cup D, V)$ ,  $D$  is a hitting set,  $|D| = l$

Candidate	Score
$w$	$8(k+1)m + 8n - 4k + 2$
$b \in B$	$8(k+1)m + 8n - 4l + 4$

If  $l \leq k$ , every  $b \in B$  will tie for first with  $w$  as the clear loser. Otherwise  $w$  will be the winner.

$(\{w\} \cup D, V)$ ,  $D$  is not a hitting set,  $|D| = l$

Candidate	Score
$w$	$8(k+1)m + 8n - 4k + 2 + 4(k+1)$
$b \in B$	$8(k+1)m + 8n - 4l + 4$

$w$  will win this election.

There is a hitting set  $B' \subset B$  where  $|B'| \leq k$  if and only if  $w$  can be made to lose the election through partition or run-off partition of candidates.

If there is a hitting set  $B' \subset B$  of size  $\leq k$ ,  $w$  can be made to lose the election through control by partition or runoff partition of candidates with the partitions  $\{w\} \cup B'$ ,  $B - B'$ .  $w$  will lose the initial subelection ( $\{w\} \cup B'$ ) as shown above, and will thus lose the entire election.

If there is no such hitting set,  $w$  cannot be made to lose the election through partition or runoff partition of candidates. As shown above,  $w$  will win any election  $(\{w\} \cup B', V)$  where  $|B'| > k$ , or where his or her opponents do not comprise a hitting set. The one special case is when  $k = n$ , where  $w$  will lose the original election but in this case there is a trivially always a hitting set, so this problem should also always accept. In any other case  $w$  will win against any subset of  $B$ , so  $w$  will win both any initial subelection and the final election, and so there is no partition to make them lose.  $\square$

## 6 Conclusions

This work leaves open a number of questions. NRV will still falls short of resistance to all cases of control, so some other natural system could still best it. Just as useful would be results about the conditions that are required for a voting system to have various resistances. It may still be that natural systems are incapable of having every control resistance simultaneously. Any useful results here would first require a formalization of what exactly a natural voting system is. Most desirable would be a reasonable set of conditions that could be shown to be incompatible with holding all resistances simultaneously, à la Arrow's Theorem.

Other useful work would be to analyze methods for sidestepping the worst-case difficulty of the control problems here. One example is the use of approximation algorithms as studied by Brelsford et al. [BFH<sup>+</sup>08], or analysis of the problems with a restricted preference-ensemble model [FHHR11]. It is important to note that the worst-case analysis performed here does not provide a guarantee that the problems will be hard on average. Still, this work provides a good first step toward understanding the behavior of these systems with respect to control.

## Acknowledgments

For helpful comments and suggestions, I am grateful to Edith Hemaspaandra, who advised my M.S. thesis in which an earlier version of part of this work appeared, Lane Hemaspaandra, Preetjot Singh, and Andrew Lin.

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